

# Binomial Option Pricing Model (single-period)

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## 1 Introduction

In this teaching note, we introduce a single-period binomial model for pricing call and put options. We assume that options are European; i.e., exercise may only occur on the expiration date. Furthermore, the underlying asset does not pay off at any time other than the expiration date; i.e., no cash flows occur prior to the expiration of the option contract.

We introduce three approaches to pricing call and put options: 1) the Delta Hedging approach, 2) the Risk Neutral Valuation approach, and 3) the Replicating Portfolio approach. Although all three approaches produce the same set of prices, they are worth studying since they shed a number of important insights into the economics of options.

## 2 Delta Hedging Approach in a Single Period

Under the delta hedging approach, we create a riskless bond by forming a perfectly hedged portfolio consisting of either a call option or a put option and the underlying stock. The price of the stock at the inception of the option contract is  $S$ , and over a discrete time interval ( $\delta t$ ) going forward, the stock will assume one of the following two values:  $S_u = uS$  or  $S_d = dS$ , where  $u > 1$  and  $d < 1$ . The probability that the stock price will move “up” is  $p$ , and the probability that the stock price will move “down” is  $1 - p$ . Throughout this teaching note, we will assume that  $S = \$100$ ,  $u = 1.05$ ,  $d = .95$ ,  $p = .60$ ,  $\delta t = 1/12$  (one month), the exercise

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price ( $K$ ) is \$100, and the interest rate ( $r$ ) is 5%. Figure 1 shows the binomial “tree” for the current (known) stock price and also the future (state-contingent) stock prices:

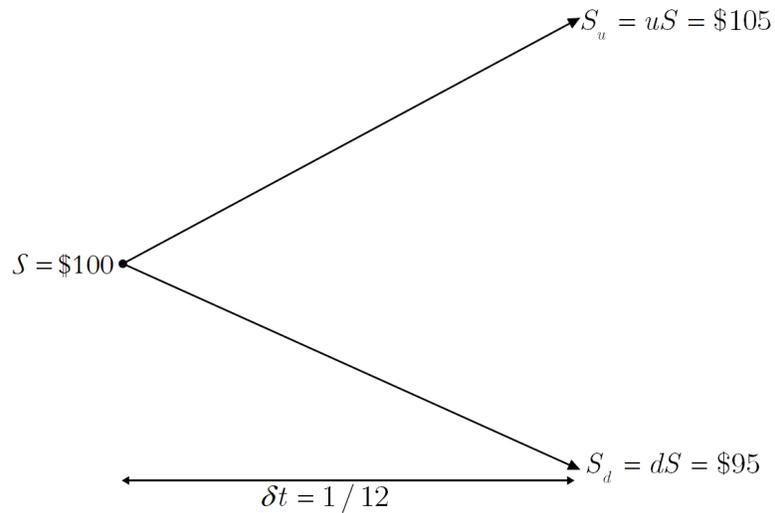


Figure 1: Single-Period Binomial Tree for the Current and Future Stock Prices

Figure 2 shows the binomial “tree” for the current (unknown) call option price and also the future (state-contingent) call option prices:

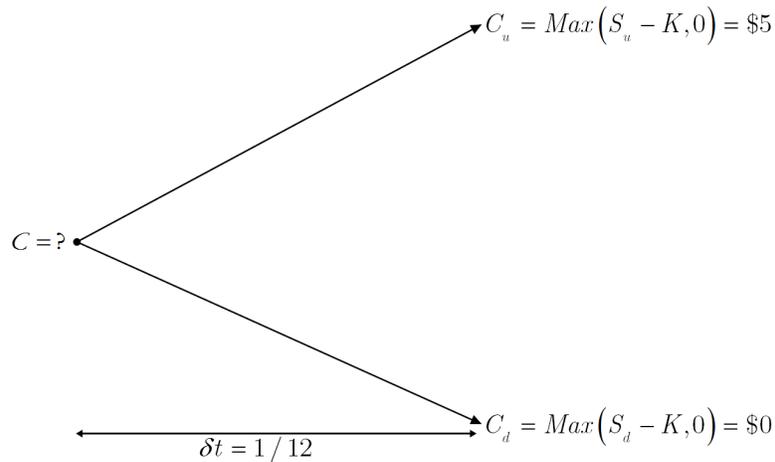


Figure 2: Single-Period Binomial Tree for the Current and Future Call Option Prices

Next, we form a “hedge” portfolio consisting of a long position in one call option and

a short position in  $\Delta$  shares of stock. This portfolio is called a hedge portfolio because movements in the value of the short stock position effectively hedge, or offset the effect of movements in the value of the long call option position. The current value of this portfolio is

$$V_H = C - \Delta S = C - \Delta 100. \quad (1)$$

At node  $u$ , the value of the hedge portfolio is equal to  $V_H^u = C_u - \Delta S_u = 5 - \Delta 105$ , and at node  $d$ , the value of the hedge portfolio is equal to  $V_H^d = C_d - \Delta S_d = 0 - \Delta 95$ . Suppose we select  $\Delta$  such that the hedge portfolio is riskless; i.e.,  $V_H^u = V_H^d$ . Solving for  $\Delta$ , we obtain:

$$V_H^u = V_H^d \Rightarrow 5 - \Delta 105 = -\Delta 95 \Rightarrow \Delta = .5. \quad (2)$$

Substituting  $\Delta = .5$  into our expressions for  $V_H^u$  and  $V_H^d$ , we obtain  $V_H^u = V_H^d = -\$47.50$ . Thus, the value of a riskless hedge portfolio comprised of one call option and a short position in .5 shares of stock is equivalent in value to a *short* position in a riskless bond. In order to prevent arbitrage, the current value of this short bond position,  $V_H = C - 50 = -\$e^{-.05/12}47.50 = -\$47.30$ , which implies that  $C = \$2.70$ .

Since we now have the fair market value for the call option, we can use the put-call parity equation to find the price of an otherwise identical put option. The put-call parity equation is given by equation (3):

$$C + Ke^{-r\delta t} = P + S. \quad (3)$$

Thus,

$$P = C + Ke^{-r\delta t} - S = \$2.70 + \$100e^{-.05/12} - \$100 = \$2.28. \quad (4)$$

We can also determine the fair market value for the put option via the delta hedging approach. Since the prices of a put option and its underlying stock are inversely related, we

form a hedge portfolio consisting of a long position in one put option and a long position in  $\Delta$  shares of stock. The current value of this portfolio is

$$V_H = P + \Delta S = P + \Delta 100. \quad (5)$$

At node  $u$ , the value of the hedge portfolio is equal to  $V_H^u = P_u + \Delta S_u = \text{Max}(K - 105, 0) + \Delta 105 = 0 + \Delta 105$ , and at node  $d$ , the value of the hedge portfolio is equal to  $V_H^d = P_d + \Delta S_d = \text{Max}(K - 95, 0) + \Delta 95 = 5 + \Delta 95$ . Suppose we select  $\Delta$  such that the hedge portfolio is riskless; i.e.,  $V_H^u = V_H^d$ . Solving for  $\Delta$ , we obtain:

$$V_H^u = V_H^d \Rightarrow \Delta 105 = 5 + \Delta 95 \Rightarrow \Delta = .5. \quad (6)$$

Substituting  $\Delta = .5$  into our expressions for  $V_H^u$  and  $V_H^d$ , we obtain  $V_H^u = V_H^d = \$52.50$ . Thus, the value of a riskless hedge portfolio comprised of one put option and a long position in .5 shares of stock is equivalent in value to a *long* position in a riskless bond. In order to prevent arbitrage, the current value of this long bond position,  $V_H = P + 50 = \$e^{-.05/12}52.50 = \$52.28$ , which implies that  $P = \$2.28$ .

### 3 Risk Neutral Valuation Approach in a Single Period

An alternative to the delta hedging approach for pricing options is the risk neutral valuation approach. As we saw in the previous section of this teaching note, since prices of call and put options and the underlying stock referenced by these options do not allow for profitable arbitrage, then it does not matter *what* we assume concerning investor risk preferences. Specifically, one can price options assuming *any type* of attitude toward risk and obtain the same answer. As such, the easiest model is the risk neutral model.

We start by showing the relationship which exists between the expected return on the

underlying stock ( $\mu$ ) and the probability of an up move ( $p = 0.60$ ). Note that

$$E(S_{\delta t}) = puS + (1 - p)dS = e^{\mu\delta t}S, \quad (7)$$

where  $E(S_{\delta t})$  represents the expected stock price one timestep from today. Solving equation (7) for  $p$ , we find that

$$p = (e^{\mu\delta t} - d) / (u - d). \quad (8)$$

Since  $p = 0.60$ , this implies that the annualized expected return on the underlying stock is  $\mu = \frac{\ln(pu + (1 - p)d)}{\delta t} = \frac{\ln(.6(1.05) + (.4).95)}{.0833} = 11.94\%$ . In a risk averse economy, investors apparently demand an (annualized) expected rate of return on the underlying stock which is 6.94 percentage points higher than the riskless rate of interest.

However, let's suppose that investors are *risk neutral*. This implies that the risk premium on the underlying asset must be 0%, not 6.94% as calculated above. Keeping this in mind, we substitute the riskless rate of interest ( $r = 5\%$ ) in place of the expected rate of return in a risk averse economy ( $\mu = 11.94\%$ ) in the numerator of equation (8) to obtain the *risk neutral* probability ( $q$ ) of an up move:

$$q = (e^{r\delta t} - d) / (u - d) = (e^{.05/12} - .95) / (.10) = .5417. \quad (9)$$

Now that we know the risk neutral probability  $q$ , we can calculate the *risk neutral expected value* of the call and put option payoffs 1 timestep (month) from now by simply weighting these payoffs by their corresponding risk neutral probabilities:

$$\hat{E}(C_{\delta t}) = qC_u + (1 - q)C_d, \text{ and} \quad (10)$$

$$\hat{E}(P_{\delta t}) = qP_u + (1 - q)P_d, \text{ where} \quad (11)$$

where  $\hat{E}()$  corresponds to the *risk neutral expected value operator*. Essentially,  $\hat{E}(C_{\delta t})$  and  $\hat{E}(P_{\delta t})$  represent the certainty-equivalent values for the call and put option payoffs at the expiration date. As such, we can determine the prices of these options by simply discounting these certainty-equivalent values for one month at the riskless rate of interest. Thus, the (arbitrage-free) prices for (single-period) European call and put options are given by the following equations:

$$C = e^{-r\delta t} \hat{E}(C_{\delta t}) = e^{-r\delta t} [qC_u + (1 - q)C_d] = e^{-.05/12} [.5417(5)] = \$2.70, \text{ and} \quad (12)$$

$$P = e^{-r\delta t} \hat{E}(P_{\delta t}) = e^{-r\delta t} [qP_u + (1 - q)P_d] = e^{-.05/12} [.4583(5)] = \$2.28. \quad (13)$$

## 4 Replicating Portfolio Approach in a Single Period

The final approach that we consider is called the replicating portfolio approach. The basic setup involves replicating the call option by buying the underlying stock on margin, and replicating the put option by lending money and shorting the underlying stock. By replicating the call and put payoffs at nodes  $u$  and  $d$ , the current market value of the replicating portfolio must equal the current market value of the option; otherwise investors can earn positive profits with zero risk and zero net investment by buying the less expensive investment and shorting the more expensive one. As in the case of the delta hedging approach, we invoke the no-arbitrage condition to establish that the price of the option must equal the value of its replicating portfolio.

We begin by computing the value of the replicating portfolio for the call option. Suppose we form a portfolio consisting of  $\Delta$  shares of stock and  $\$B$  in riskless bonds. The initial cost of forming such a portfolio is  $\$(\Delta S + B)$ . One timestep from now,

$$C_u = \Delta uS + e^{r\delta t} B, \text{ and} \quad (14)$$

$$C_d = \Delta dS + e^{r\delta t}B. \quad (15)$$

Thus, we have two equations in two unknowns. Solving equations (14) and (15) for  $\Delta$  and  $B$ , we obtain:<sup>1</sup>

$$\Delta = \frac{C_u - C_d}{uS - dS} \geq 0, \text{ and} \quad (16)$$

$$B = \frac{uC_d - dC_u}{e^{r\delta t}(u - d)} \leq 0. \quad (17)$$

Next, let's reconsider these equations in light of our numerical example. From equations (16) and (17),  $\Delta = \frac{C_u - C_d}{uS - dS} = 5/10 = .5$  and  $B = \frac{uC_d - dC_u}{e^{r\delta t}(u - d)} = \frac{1.05(0) - .95(5)}{e^{.05/12}(.1)} = -47.30$ . Thus, we can replicate the call option by purchasing half of a share for \$50 and borrowing \$47.30. Since the value of the replicating portfolio is  $\$(\Delta S + B) = \$50 - 47.30 = \$2.70$ , this must also be the value of the call option.

Next, we compute the value of the replicating portfolio for the put option. Suppose we form a portfolio consisting of  $\Delta$  shares of stock and  $\$B$  in riskless bonds. The initial cost of forming such a portfolio is  $\$(\Delta S + B)$ . One timestep from now,

$$P_u = \Delta uS + e^{r\delta t}B, \text{ and} \quad (18)$$

$$P_d = \Delta dS + e^{r\delta t}B. \quad (19)$$

Solving equations (18) and (19) for  $\Delta$  and  $B$ , we obtain:

$$\Delta = \frac{P_u - P_d}{uS - dS} \leq 0, \text{ and} \quad (20)$$

$$B = \frac{uP_d - dP_u}{e^{r\delta t}(u - d)} \geq 0. \quad (21)$$

Next, let's reconsider these equations in light of our numerical example. From equations

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<sup>1</sup>An appendix which details the mathematical logic behind equations (16) and (17) and equations (20) and (21) is available upon request.

(20) and (21),  $\Delta = \frac{P_u - P_d}{uS - dS} = -5/10 = -.5$  and  $B = \frac{uP_d - dP_u}{e^{r\delta t}(u - d)} = \frac{1.05(5) - .95(0)}{e^{.05/12}(.1)} = 52.28$ . Thus, we can replicate the put option by shorting half of a share for -\$50 and lending \$52.28. Since the value of the replicating portfolio is  $\$(\Delta S + B) = -\$50 + \$52.28 = \$2.28$ , this must also be the value of the put option.